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Dirac operators on Lagrangian submanifolds☆

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Abstract

We study a natural Dirac operator on a Lagrangian submanifold of a Kähler manifold. We first show that its square coincides with the Hodge–de Rham Laplacian provided the complex structure identifies the spin structures of the tangent and normal bundles of the submanifold. We then give extrinsic estimates for the eigenvalues of that operator and discuss some examples.

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1. Introduction

The main object of this paper is to initiate the study of the properties of a Dirac operator on Lagrangian submanifolds of Kähler manifolds.

Spin geometry has revealed as a powerful tool in intrinsic geometry for a long time (see e.g. [18]). It is however a recent and striking fact that spinors play a role in extrinsic geometry as well. Initiated by Witten [26], the use of Dirac operators on submanifolds has only been developed over the last years, especially about the following question: how can one relate analytical properties of some Dirac operators on a submanifold with extrinsic geometric quantities? For submanifolds of real space-forms, on which there exists particular spinor fields (mainly parallel spinor fields, up to a conformal change of the metric), a beautiful series of results has already appeared (see [10] for references). However, answering that question in presence of further geometric structures seems to have been little considered.

☆ This work is partially based on the Ph.D. thesis of the author [10].

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We propose in this paper to begin with the study of (immersed) submanifolds of *Kählerian* manifolds. The presence of a complex structure on the ambient manifold gives rise to a rich variety of submanifolds (totally real, Kählerian, real hypersurface, etc.). That is why we shall restrict our attention to a particular class of submanifolds, namely *Lagrangian* submanifolds. A submanifold of a Kählerian manifold is Lagrangian if and only if the (ambient) complex structure maps its tangent bundle *onto* its normal bundle. Like every submanifold, a Lagrangian submanifold carries a *twisted-Dirac operator*. We shall first prove that, if furthermore the complex structure identifies the spin structures of the tangent and normal bundles, then this twisted-Dirac operator identifies with the *Euler* operator. This requires adapting some technical algebraic Lemmas (compare with [2,12]), which we shall therefore recall in detail in the first part. Coming back to the original question, we then prove new eigenvalue estimates for the above twisted-Dirac operator, and show their sharpness through examples. The results obtained show analogies with [6,24].

2. Spin structures and Dirac operators on a Lagrangian submanifold

We begin with collecting basic facts about spin structures on Lagrangian submanifolds of Kählerian manifolds (see also [5,7,18] for general spin geometry). We first describe the necessary algebraic material, then transport it to bundles with the help of a group-equivariance condition.

2.1. Clifford algebras and spinors

In this subsection, we recall some important isomorphisms between the complex Clifford algebra (see definition below) and other vector spaces. We point out that the isomorphism (7) below slightly differs from the equivalent one in [12] or [2], since we want here to keep track of the "Clifford action" in a more suitable way for our setting.

We fix a positive integer n, and denote by "can" the standard Euclidean inner product of $\mathbb{R}^n$. Throughout this paper, unless explicitly mentioned, all the isomorphisms will be denoted by the identity map.

Let $\mathbb{C}\mathrm{l}_n$ (resp. Cl_n) be the complex (resp. real) Clifford algebra of $(\mathbb{R}^n, \mathrm{can})$, that is, the only associative complex (resp. real) algebra with unit generated by $\mathbb{R}^n$ with the relation

$$v \cdot w + w \cdot v = -2\,\mathrm{can}(v, w)1$$

for all vectors v and w in $\mathbb{R}^n$. Here the product of $\mathbb{C}\mathrm{l}_n$ (resp. Cl_n), denoted "$\cdot$", is called the *Clifford multiplication*. We recall the properties of $\mathbb{C}\mathrm{l}_n$ which will be important for the future:

- Let $\Lambda\mathbb{R}^n \otimes \mathbb{C}$ be the complexified exterior algebra of $\mathbb{R}^n$. Then there exists a *canonical* linear isomorphism [18]

$$\mathbb{C}\mathrm{l}_n \to \Lambda\mathbb{R}^n \otimes \mathbb{C}, \tag{1}$$

  which maps every element of the form $v \cdot \varphi$ ($v \in \mathbb{R}^n$, $\varphi \in \mathbb{C}\mathrm{l}_n$) onto $v \wedge \varphi - v \lrcorner \varphi$, where "$v \lrcorner \varphi$" stands for $v^\flat \lrcorner \varphi$ through the musical isomorphism $v \mapsto v^\flat := \mathrm{can}(v, \cdot)$ between $\mathbb{R}^n$ and $(\mathbb{R}^n)^*$. We hereby identify through (1) the space $\Lambda^p\mathbb{R}^n \otimes \mathbb{C}$ as a subspace of $\mathbb{C}\mathrm{l}_n$.

- The algebra $\mathbb{C}l_n$ is either a complex matrix algebra or the copy of two such ones: there exists a complex vector space Σ_n of dimension $2^{[n/2]}$, called the *space of spinors*, and an isomorphism of complex algebras:

$$\mathbb{C}l_n \cong \begin{cases} \mathrm{End}_{\mathbb{C}}(\Sigma_n) & \text{if } n \text{ is even}, \\ \mathrm{End}_{\mathbb{C}}(\Sigma_n) \oplus \mathrm{End}_{\mathbb{C}}(\Sigma_n) & \text{if } n \text{ is odd}. \end{cases} \tag{2}$$

Without loss of generality (see e.g. [18]), we further assume that, when n is odd, the isomorphism (2) maps the complex volume element of $\mathbb{C}l_n$ (see e.g. in [5] for its definition) onto $\mathrm{Id}_{\Sigma_n} \oplus -\mathrm{Id}_{\Sigma_n}$. We define δ_n as the isomorphism (2) if n is even, and the composition of the projection onto the first subalgebra $\mathrm{End}_{\mathbb{C}}(\Sigma_n)$ with (2) if n is odd. In particular, for n odd and for every v in $\mathbb{R}^n$, the isomorphism (2) reads $v \mapsto \delta_n(v) \oplus -\delta_n(v)$, see [18].

- The space Σ_n carries a natural Hermitian inner product "$\langle \cdot, \cdot \rangle$" (which we assume to be complex-linear in the *first* argument) such that, for every v in $\mathbb{R}^n$ and all σ, σ' in Σ_n:

$$\langle \delta_n(v)\sigma, \sigma' \rangle = -\langle \sigma, \delta_n(v)\sigma' \rangle. \tag{3}$$

The property (3) determines the Hermitian inner product "$\langle \cdot, \cdot \rangle$" up to a positive scalar [5].

Define now the *spin group* Spin_n as

$$\mathrm{Spin}_n := \{v_1 \cdots v_{2k} / k \geq 1, v_j \in \mathbb{R}^n, \mathrm{can}(v_j, v_j) = 1\}$$

and the *spin representation* to be the restriction of δ_n to Spin_n. The spin group is a compact Lie-subgroup of the group of invertible elements in $\mathbb{C}l_n$ which has the following remarkable properties:

- There exists a two-fold covering Lie-group-homomorphism from Spin_n onto the special orthogonal group SO_n, which we denote by "Ad".
- Denoting also "Ad" the composition of the natural representation of SO_n on $\Lambda\mathbb{R}^n \otimes \mathbb{C}$ with Ad, the isomorphism (1) is Spin_n-equivariant, i.e., for every u in Spin_n and φ in $\mathbb{C}l_n$,

$$u \cdot \varphi \cdot u^{-1} \simeq \mathrm{Ad}(u)\varphi$$

through (1).

- Every Hermitian inner product "$\langle \cdot, \cdot \rangle$" satisfying (3) is Spin_n-invariant, i.e., the spin representation is unitary w.r.t. "$\langle \cdot, \cdot \rangle$".

We now recall two lemmas and discuss their consequences.

Lemma 1. *There exists a complex-antilinear automorphism $\jmath$ of Σ_n commuting with the spin representation, i.e., for every u in* Spin_n, *we have* $\delta_n(u) \circ \jmath = \jmath \circ \delta_n(u)$.

Proof. Although it follows from representation theory (see e.g. p. 21 in [25] or Section 1.7 in [7]), we give here an elementary argument.

From the classification of *real* Clifford algebras (see [18]), we have:

$$\mathrm{Cl}_n \cong \begin{cases} \mathbb{R}(2^{[n/2]}) & \text{if } n \equiv 0 \text{ or } 6 \quad (8), \\ \mathbb{R}(2^{[n/2]}) \oplus \mathbb{R}(2^{[n/2]}) & \text{if } n \equiv 7 \quad (8), \\ \mathbb{H}(2^{[(n-1)/2]}) & \text{if } n \equiv 2 \text{ or } 4 \quad (8), \\ \mathbb{H}(2^{[n-2/2]}) \oplus \mathbb{H}(2^{[n-2/2]}) & \text{if } n \equiv 3 \quad (8), \\ \mathbb{C}(2^{[n/2]}) & \text{if } n \equiv 1 \text{ or } 5 \quad (8). \end{cases}$$

As $\mathbb{C}\mathrm{l}_n \cong \mathrm{Cl}_n \otimes \mathbb{C}$, we see that, if $n \equiv 0$, 6 or 7 (8), the *complex* representation $\delta_n : \mathbb{C}\mathrm{l}_n \to \mathrm{End}_{\mathbb{C}}(\Sigma_n)$ admits a real structure, i.e., there exists a $\mathbb{C}$-antilinear and involutive automorphism $\jmath$ of Σ_n such that, for every vector v in $\mathbb{R}^n$ (hence for every element in $\mathbb{C}\mathrm{l}_n$),

$$\delta_n(v) \circ \jmath = \jmath \circ \delta_n(v).$$

If $n \equiv 2$, 3 or 4 (8), there exists a quaternionic structure on Σ_n, i.e., a $\mathbb{C}$-antilinear automorphism $\jmath$ of Σ_n satisfying $\jmath^2 = -\mathrm{Id}$ and the preceding relation. If $n \equiv 1$ or 5 (8), the real representation of Cl_n being already complex, there exists no $\mathbb{C}$-antilinear automorphism of Σ_n commuting with the action of *every* vector of $\mathbb{R}^n$ as before. However, the relation we look for needs only to hold on Spin_n and not on Cl_n. We eliminate the obvious case $n = 1$. For $n > 1$, as Spin_n is a subset of $\mathrm{Cl}_n^0 := \oplus_{p\,\text{even}} \Lambda^p \mathbb{R}^n$, and Cl_n^0 identifies with Cl_{n-1} through an algebra-isomorphism which provides the equivalence of δ_{n-1} (or "double copy" as in (2)) with $(\delta_n)_{|_{\mathrm{Cl}_n^0}}$ (see [18]), we just need to solve the problem for δ_{n-1}. But from the preceding arguments, the representation δ_{n-1} admits a real or quaternionic structure, so that we again obtain a $\mathbb{C}$-antilinear automorphism $\jmath$ of Σ_n which commutes with $(\delta_n)_{|\mathrm{Spin}_n}$.

To sum up, for every $n \geq 1$, there exists a $\mathbb{C}$-antilinear automorphism $\jmath$ of Σ_n such that, for every u in Spin_n,

$$\delta_n(u) \circ \jmath = \jmath \circ \delta_n(u),$$

which is the desired property. □

Corollary 1 (see e.g. [9] or p. 244 in [1]). *There exists a complex-linear isomorphism*

$$\mathbb{C}\mathrm{l}_n \to \begin{cases} \Sigma_n \otimes \Sigma_n & \text{if } n \text{ is even}, \\ \Sigma_n \otimes \Sigma_n \oplus \Sigma_n \otimes \Sigma_n & \text{if } n \text{ is odd} \end{cases} \tag{4}$$

satisfying:

- *For every v in $\mathbb{R}^n$ and every φ in $\mathbb{C}\mathrm{l}_n$, the element $v \cdot \varphi$ is mapped onto $\{\delta_n(v) \otimes \mathrm{Id}\}\varphi$ when n is even (resp. onto $\{\delta_n(v) \otimes \mathrm{Id} \oplus -\delta_n(v) \otimes \mathrm{Id}\}\varphi$ when n is odd).*
- *The isomorphism (4) is Spin_n-equivariant: for every u in Spin_n and every φ in $\mathbb{C}\mathrm{l}_n$, the element $u \cdot \varphi \cdot u^{-1}$ is mapped onto $\{\delta_n(u) \otimes \delta_n(u)\}\varphi$ when n is even (resp. onto $\{\delta_n(u) \otimes \delta_n(u) \oplus \delta_n(u) \otimes \delta_n(u)\}\varphi$ when n is odd).*

Proof. The proof obviously follows from the preceding lemma. □

For the next lemma, we recall an explicit description of the space of spinors as a subspace of the complex Clifford algebra in even dimensions (compare with [5,16,17]). We consider

$\mathbb{R}^{2n}$ endowed with its natural complex structure J, so that $\mathbb{R}^{2n} = \mathbb{R}^n \oplus J(\mathbb{R}^n)$. Let $p_\pm$ the two projectors of $\mathbb{R}^{2n} \otimes \mathbb{C}$ defined by

$$p_\pm := \frac{1}{2}(\mathrm{Id} \mp iJ),$$

where J is extended as a complex-linear automorphism of $\mathbb{R}^{2n} \otimes \mathbb{C}$. The endomorphisms p_+ and p_- satisfy the following: $p_- \circ p_+ = p_+ \circ p_- = 0$, $p_\pm \circ J = J \circ p_\pm = \pm i p_\pm$, and $\mathrm{can}(p_+(Z), Z') = \mathrm{can}(Z, p_-(Z'))$ for all Z, Z' in $\mathbb{R}^{2n} \otimes \mathbb{C}$.

Let $(e_j)_{1 \le j \le n}$ be the canonical basis of $\mathbb{R}^n$. For $1 \le j \le n$, define $z_j := p_+(e_j)$ and $\bar{z}_j := p_-(e_j)$. The vectors $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n$ form the so-called *Witt-basis* of $\mathbb{R}^{2n} \otimes \mathbb{C}$ associated to the basis $(e_j)_{1 \le j \le n}$ of $\mathbb{R}^n$. Set

$$\bar{\omega} := \bar{z}_1 \cdots \bar{z}_n.$$

From the above properties of $p_\pm$, that element of Cl_{2n} is independent of the choice of the positively oriented orthonormal basis (p.o.n.b.) of $\mathbb{R}^n$: replacing $(e_j)_{1 \le j \le n}$ by another p.o.n.b. of $\mathbb{R}^n$, and taking the associated Witt-basis of $\mathbb{R}^{2n} \otimes \mathbb{C}$, one obtains the same element $\bar{\omega}$.

For $1 \le p \le n$, set $\mathcal{L}^p := \mathrm{Span}_\mathbb{C}\{z_{i_1} \cdots z_{i_p} \cdot \bar{\omega}, 1 \le i_1 < \cdots < i_p \le n\}$ and $\mathcal{L}^0 := \mathbb{C}\bar{\omega}$. Those subspaces of Cl_{2n} do not depend on the choice of a p.o.n.b. of $\mathbb{R}^n$, in the preceding sense. It can furthermore be shown that $\oplus_{p=0}^n \mathcal{L}^p$ is a left-ideal of dimension 2^n in Cl_{2n}, hence is isomorphic to Σ_{2n} (see [5]). We can then set

$$\Sigma_{2n} := \bigoplus_{p=0}^{n} \mathcal{L}^p.$$

Note here that $\dim_\mathbb{C}(\mathcal{L}^p) = C_n^p := n!/p!(n-p)!$. Through that identification, for each ψ in Cl_{2n}, the endomorphism $\delta_{2n}(\psi)$ is given by the left-Clifford multiplication by ψ. For example (and this will be crucial for the future), the Clifford multiplication by the Kähler form $\tilde{\Omega}(\cdot,\cdot) = \mathrm{can}(J\cdot,\cdot)$ of $(\mathbb{R}^{2n}, J)$ is given by

$$\delta_{2n}(\tilde{\Omega}) = \bigoplus_{p=0}^{n} i(2p-n)\mathrm{Id}_{\mathcal{L}^p},$$

that is, $\mathcal{L}^p$ is the eigenspace of $\delta_{2n}(\tilde{\Omega})$ for the eigenvalue $i(2p-n)$.

Moreover, a Hermitian inner product satisfying (3) can be defined in the following way: for any $1 \le i_1 < \cdots < i_p \le n$ and $1 \le j_1 < \cdots < j_q \le n$, set

$$\langle z_{i_1} \cdots z_{i_p} \cdot \bar{\omega}, z_{j_1} \cdots z_{j_q} \cdot \bar{\omega} \rangle := \begin{cases} 0 & \text{if } \{i_1, \ldots, i_p\} \neq \{j_1, \ldots, j_q\}, \\ 2^{[(n+1)/2]} & \text{otherwise.} \end{cases}$$

It can be shown that (3) holds and that this Hermitian inner product does not depend on the choice of a p.o.n.b. of $\mathbb{R}^n$.

We furthermore define Spin'_n to be the spin group of $(J\mathbb{R}^n, \mathrm{can})$, i.e.,

$$\mathrm{Spin}'_n := \{w_1 \cdots w_{2k} / k \ge 1, w_j \in J\mathbb{R}^n, \mathrm{can}(w_j, w_j) = 1\} \subset \mathrm{Cl}_{2n}.$$

From the universal property of Clifford algebras [18], the linear isometry $J : \mathbb{R}^n \to J\mathbb{R}^n$ induces a Lie-group-isomorphism $\tilde{J} : \mathrm{Spin}_n \to \mathrm{Spin}'_n$. We then set $\delta'_n := \delta_{n|_{\mathrm{Spin}_n}} \circ (\tilde{J})^{-1} : \mathrm{Spin}'_n \to \mathrm{Aut}_\mathbb{C}(\Sigma_n)$. Note that elements of Spin_n and Spin'_n obviously commute.

Lemma 2. *Consider $\mathbb{C}l_n$ as canonically embedded in $\mathbb{C}l_{2n}$. Then the map*

$$\mathbb{C}l_n \to \Sigma_{2n}, \qquad \varphi \mapsto \varphi \cdot \bar{\omega} \tag{5}$$

is a complex-linear isomorphism satisfying:

- *For every v in $\mathbb{R}^n$ and every φ in $\mathbb{C}l_n$, the element $v \cdot \varphi$ is mapped onto $v \cdot \varphi \cdot \bar{\omega}$.*
- *For every w in $J\mathbb{R}^n$ and every φ in $\Lambda^p\mathbb{R}^n \otimes \mathbb{C}$, the element $(-1)^{p+1}i\varphi \cdot J(w)$ is mapped onto $w \cdot \varphi \cdot \bar{\omega}$.*

In particular, the isomorphism (5) is Spin_n*-equivariant w.r.t. the* "*diagonal immersion*"

$$\mathrm{Spin}_n \to \mathrm{Spin}_{2n}, \qquad u \mapsto u \cdot \tilde{J}(u), \tag{6}$$

that is, for every u in Spin_n *and φ in $\mathbb{C}l_n$, the element $u \cdot \varphi \cdot u^{-1}$ is mapped onto $u \cdot \tilde{J}(u) \cdot \varphi \cdot \bar{\omega}$.*

Proof. Since the linear map (5) is obviously surjective (for each $1 \le p \le n$ and each $1 \le i_1 < \cdots < i_p \le n$, the element $e_{i_1} \cdots e_{i_p}$ of $\mathbb{C}l_n$ is mapped onto $z_{i_1} \cdot \ldots z_{i_p} \cdot \bar{\omega}$), and both spaces have the same dimension, it is a linear isomorphism. Furthermore, (5) maps the subspace $\Lambda^p\mathbb{R}^n \otimes \mathbb{C}$ onto $\mathcal{L}^p$.

The first property is trivial. On the other hand, for every w in $J\mathbb{R}^n$ and φ in $\Lambda^p\mathbb{R}^n \otimes \mathbb{C}$,

$$\begin{aligned}(-1)^{p+1}i\varphi \cdot J(w) \cdot \bar{\omega} &= (-1)^p\varphi \cdot (-ip_+(J(w)) \cdot \bar{\omega}) \\ &= (-1)^p\varphi \cdot p_+(w) \cdot \bar{\omega} = (-1)^p\varphi \cdot w \cdot \bar{\omega} = w \cdot \varphi \cdot \bar{\omega},\end{aligned}$$

hence the second point holds. As a consequence, for all w_1, w_2 in $J\mathbb{R}^n$ with $\mathrm{can}(w_1, w_1) = \mathrm{can}(w_2, w_2) = 1$ and every φ in $\mathcal{L}^p$, the preimage through (5) of $w_1 \cdot w_2 \cdot \varphi \cdot \bar{\omega}$ is given by

$$\begin{aligned}(-1)^p i\{w_2 \cdot \varphi\} \cdot J(w_1) \cdot \bar{\omega} &= (-1)^p(-1)^{p+1}i^2\varphi \cdot J(w_2) \cdot J(w_1) \cdot \bar{\omega} \\ &= \varphi \cdot J(w_2) \cdot J(w_1) \cdot \bar{\omega},\end{aligned}$$

i.e., is equal to $\varphi \cdot J(w_2) \cdot J(w_1)$. Note that it does no longer depend on p. Since $J(w_2) \cdot J(w_1) = (\tilde{J})^{-1}(w_2 \cdot w_1) = (\tilde{J})^{-1}\{(w_1 \cdot w_2)^{-1}\}$, we obtain

$$\varphi \cdot (\tilde{J})^{-1}(u'^{-1}) \overset{(5)}{\simeq} u' \cdot \varphi \cdot \bar{\omega}$$

for every u' in Spin_n' and every φ in $\mathbb{C}l_n$, from which follows the last statement. □

Corollary 2. *There exists a complex-linear isomorphism*

$$\Sigma_{2n} \to \begin{cases} \Sigma_n \otimes \Sigma_n & \text{if } n \text{ is even}, \\ \Sigma_n \otimes \Sigma_n \oplus \Sigma_n \otimes \Sigma_n & \text{if } n \text{ isodd}, \end{cases} \tag{7}$$

satisfying:

- *For every v in $\mathbb{R}^n$ and φ in Σ_{2n}, the element $\delta_{2n}(v)\varphi$ is mapped onto $\{\delta_n(v) \otimes \mathrm{Id}\}\varphi$ if n is even* (*resp. onto $\{\delta_n(v) \otimes \mathrm{Id} \oplus -\delta_n(v) \otimes \mathrm{Id}\}\varphi$ if n is odd*).
- *For every w in $J\mathbb{R}^n$ and φ* in *$\mathcal{L}^p$, the element $\delta_{2n}(w)\varphi$ is mapped onto $(-1)^p i\{\mathrm{Id} \otimes \delta_n(J(w))\}\varphi$ if n is even* (*resp. onto $(-1)^p i\{\mathrm{Id} \otimes \delta_n(J(w)) \oplus -\mathrm{Id} \otimes \delta_n(J(w))\}\varphi$ if n is odd*).

- *If $\Sigma_n \otimes \Sigma_n$ is endowed with the tensor product of a Hermitian inner product satisfying (3) with itself, then the isomorphism (7) is unitary.*

In particular, the inverse of (7) is $\mathrm{Spin}_n \times \mathrm{Spin}'_n$-equivariant w.r.t. the group-homomorphism

$$\mathrm{Spin}_n \times \mathrm{Spin}'_n \to \mathrm{Spin}_{2n}, \qquad (u, u') \mapsto u \cdot u',$$

that is, for every (u, u') in $\mathrm{Spin}_n \times \mathrm{Spin}'_n$ and every φ in Σ_{2n}, the isomorphism (7) maps the element $\delta_{2n}(u \cdot u')\varphi$ onto $\{\delta_n(u) \otimes \delta'_n(u')\}\varphi$ if n is even (resp. onto $\{\delta_n(u) \otimes \delta'_n(u') \oplus \delta_n(u) \otimes \delta'_n(u')\}\varphi$ if n is odd).

Proof. The isomorphism (7) is obtained bringing together the isomorphisms (4) and (5), and straightforward satisfies the first property. The second one follows from Lemma 2 and from

$$\varphi \cdot v \overset{(4)}{=} \begin{cases} -\{\mathrm{Id} \otimes \delta_n(v)\}\varphi & \text{if } n \text{ iseven} \\ -\{\mathrm{Id} \otimes \delta_n(v) \oplus -\mathrm{Id} \otimes \delta_n(v)\}\varphi & \text{if } n \text{ is odd.} \end{cases}$$

for every v in $\mathbb{R}^n$ and every φ in $\mathbb{C}\mathrm{l}_n$. The third one comes from the fact that the squared-norm of the image of $e_{i_1} \cdots e_{i_p} \cdot \bar{\omega}$ is $2^{[(n+1)/2]}$ (remember that $(e_j)_{1 \le j \le n}$ stands here for the canonical basis of $\mathbb{R}^n$). The last statement follows from the two first ones, since for all vectors w_1 and w_2 in $J\mathbb{R}^n$, the isomorphism (7) maps $\delta_{2n}(w_1)\delta_{2n}(w_2)$ onto $\mathrm{Id} \otimes \{\delta_n(J(w_1))\delta_n(J(w_2))\}$ if n is even (resp. onto $\mathrm{Id} \otimes \{\delta_n(J(w_1))\delta_n(J(w_2))\} \oplus \mathrm{Id} \otimes \{\delta_n(J(w_1))\delta_n(J(w_2))\}$ if n is odd). □

Corollary 3. *There exists a complex-linear isomorphism*

$$\Sigma_{2n} \to \Lambda\mathbb{R}^n \otimes \mathbb{C} \tag{8}$$

satisfying:

- *For every v in $\mathbb{R}^n$ and φ in Σ_{2n}, the element $\delta_{2n}(v)\varphi$ is mapped onto $v \wedge \varphi - v \lrcorner \varphi$.*
- *For every w in $J\mathbb{R}^n$ and φ in Σ_{2n}, the element $\delta_{2n}(w)\varphi$ is mapped onto $-i\{J(w) \wedge \varphi + J(w) \lrcorner \varphi\}$.*

In particular, the isomorphism (8) is Spin_n-equivariant w.r.t. (6), i.e., for every u in Spin_n and every φ in Σ_{2n}, the isomorphism (8) maps $u \cdot \tilde{J}(u) \cdot \varphi$ onto $\mathrm{Ad}(u)\varphi$.

Proof. As before, the isomorphism (8) is obtained from the isomorphisms (1) and (5), and satisfies the first property. The second statement comes from the fact that, for every vector v in $\mathbb{R}^n$ and every φ in $\Lambda^p\mathbb{R}^n \otimes \mathbb{C}$, the element $\varphi \cdot v$ corresponds through (1) to the form $(-1)^p\{v \wedge \varphi + v \lrcorner \varphi\}$ (see [18]). The last one straightforward follows from the preliminary remarks and Lemma 2. □

2.2. Spinor bundles on a Lagrangian submanifold

We now deduce from the first subsection isomorphisms between spinor bundles on a submanifold. Since we shall need those isomorphisms in the setting of Lagrangian submanifolds (see definition below), we restrict to the case of a Riemannian submanifold (M^n, g) of dimension n immersed in a Riemannian manifold $(\tilde{M}^{2n}, g)$ of (real) dimension $2n$. We shall always use the following notations: II will be the bundle-valued second fundamental form of the immersion, H its mean-curvature vector field (in our convention, $H := 1/n\,\mathrm{tr}(II)$), and ∇^M (resp. $\tilde{\nabla}$) will be the Levi-Cività connection of (M, g) (resp. of $(\tilde{M}, g)$). The induced covariant derivative on the exterior bundle will be denoted analogously.

We assume $\tilde{M}$ to be spin, and fix a spin structure $\mathrm{Spin}(T\tilde{M}) \to \mathrm{SO}(T\tilde{M})$. As it is in general impossible to induce a spin structure from $\tilde{M}$ to M (compare with the case of an oriented hypersurface [2,18,22]), we assume M to be spin as well and fix a spin structure $\mathrm{Spin}(TM) \to \mathrm{SO}(TM)$ on M. Then the normal bundle NM of M in $\tilde{M}$ is spin, and carries an induced spin structure, $\mathrm{Spin}(NM) \to \mathrm{SO}(NM)$, for which there exists a principal-bundle-homomorphism $\mathrm{Spin}(TM) \times_M \mathrm{Spin}(NM) \to \mathrm{Spin}(T\tilde{M})_{|M}$ making the following diagram commutative [21]:

$$\begin{tikzcd}
\mathrm{Spin}(TM)\times_M \mathrm{Spin}(NM) \arrow[r] \arrow[dd] & \mathrm{Spin}(T\widetilde{M})_{|M} \arrow[dd] \arrow[dr] & \\
& & M \\
\mathrm{SO}(TM)\times_M \mathrm{SO}(NM) \arrow[r] & \mathrm{SO}(T\widetilde{M})_{|M} \arrow[ur] &
\end{tikzcd}$$

Let ΣM (resp. ΣN, $\Sigma\tilde{M}$) be the spinor bundle of TM (resp. of NM, $T\tilde{M}$), i.e., the complex vector bundle associated to the spin bundle through the spin representation. There are three fundamental objects on the spinor bundle:

- The isomorphism (2) being obviously Spin_n-equivariant induces a bilinear map, called the *Clifford multiplication*

$$TM \times_M \Sigma M \to \Sigma M, \qquad (X, \varphi) \mapsto \gamma_M(X)\varphi$$

satisfying

$$\gamma_M(X)\gamma_M(Y) + \gamma_M(Y)\gamma_M(X) = -2g(X, Y)\mathrm{Id}_{\Sigma M}$$

for all vectors X and Y in TM. The same holds for ΣN and $\Sigma\tilde{M}$; we denote by $X \cdot \varphi := \gamma_{\tilde{M}}(X)\varphi$ the Clifford multiplication by a vector X on an element φ of $\Sigma\tilde{M}$.
- The spinor bundle ΣM also inherits from the space of spinors a Hermitian inner product "$\langle\cdot,\cdot\rangle_M$" satisfying

$$\langle\gamma_M(X)\varphi, \psi\rangle_M = -\langle\varphi, \gamma_M(X)\psi\rangle_M$$

for every X in TM and all φ, ψ in ΣM. The same property holds for ΣN and for $\Sigma\tilde{M}$, for which such a Hermitian inner product will be denoted by "$\langle\cdot,\cdot\rangle$".
- The Levi-Cività connection of (TM, g) induces a covariant derivative $\nabla^{\Sigma M}$ on ΣM [5,18]. This covariant derivative is metric w.r.t. "$\langle\cdot,\cdot\rangle_M$" and satisfies the Leibniz rule w.r.t. the Clifford multiplication. We denote by $\nabla^{\Sigma N}$ (resp. $\tilde{\nabla}$) that covariant derivative on ΣN (resp. on $\Sigma\tilde{M}$).

We now compare the different spinor bundles on the submanifold M. We need further notations in that purpose. For a tangent vector X to M and an element ϕ of $\Sigma M \otimes \Sigma N$ if n is even (resp. of $\Sigma M \otimes \Sigma N \oplus \Sigma M \otimes \Sigma N$ if n is odd), we define "$X \cdot_M \phi$" to be

$$\left|\begin{array}{ll} \{\gamma_M(X) \otimes \mathrm{Id}_{\Sigma N}\}\phi & \text{if } n \text{ is even} \\ \{\gamma_M(X) \otimes \mathrm{Id}_{\Sigma N} \oplus -\gamma_M(X) \otimes \mathrm{Id}_{\Sigma N}\}\phi & \text{if } n \text{ is odd.} \end{array}\right.$$

We furthermore set

$$\nabla := \left|\begin{array}{ll} \nabla^{\Sigma M \otimes \Sigma N} & \text{if } m \text{ or } n \text{ is even} \\ \nabla^{\Sigma M \otimes \Sigma N} \oplus \nabla^{\Sigma M \otimes \Sigma N} & \text{otherwise.} \end{array}\right.$$

Note that ∇ is *not* the natural covariant derivative of ΣM, since from its definition it depends on the covariant derivative of the normal bundle.

From the above homomorphism between spin bundles and the preceding subsection, we have the following:

Lemma 3. *There exists a complex-vector bundle isomorphism*

$$\Sigma\tilde{M}_{|M} \to \begin{cases} \Sigma M \otimes \Sigma N & \text{if } n \text{ is even,} \\ \Sigma M \otimes \Sigma N \oplus \Sigma M \otimes \Sigma N & \text{if } n \text{ is odd} \end{cases} \tag{9}$$

satisfying:

- *For every tangent vector field X on M and every section ϕ of $\Sigma\tilde{M}_{|M}$, the isomorphism (9) maps the section $X \cdot \phi$ onto $X \cdot_M \phi$.*
- *For every tangent vector field X on M and every section ϕ of $\Sigma\tilde{M}_{|M}$,*

$$\tilde{\nabla}_X\phi = \nabla_X\phi + \frac{1}{2}\sum_{j=1}^{n} e_j \cdot II(X, e_j) \cdot \phi$$

in any local o.n.b. $(e_j)_{1\le j\le n}$ of TM.

Furthermore, the isomorphism (9) can be assumed to be unitary.

Proof. From its equivariance under the action of $\mathrm{Spin}_n \times \mathrm{Spin}'_n$, the isomorphism (7) straightforward induces the isomorphism (9) between the vector bundles. The first property is just the translation of that of (7) on vector bundles. The second one is deduced in a quite analogous way as in [12] from the three following points: use the local expressions of the covariant derivatives $\tilde{\nabla}$ and ∇ [5,18], apply the Gauß-Weingarten formula on $T\tilde{M}_{|M}$, and use the correspondence through (9) between the Clifford multiplications by 2-forms, that is: for all vectors X_1 and X_2 in TM,

$$X_1 \cdot X_2 \cdot \overset{(9)}{\simeq} X_1 \underset{M}{\cdot} X_2 \underset{M}{\cdot},$$

and for all vectors ν_1, ν_2 in NM,

$$\nu_1 \cdot \nu_2 \cdot \overset{(9)}{\simeq} \mathrm{Id} \otimes \gamma_N(\nu_1)\gamma_N(\nu_2)(\oplus\, \mathrm{Id} \otimes \gamma_N(\nu_1)\gamma_N(\nu_2)),$$

where the parentheses stand for the case "n odd" (see previous subsection). The last remark is also a direct consequence of Corollary 2. □

In this setup, the most natural Dirac operators that can be introduced on the manifold M are the so-called *twisted-Dirac operator* $D_M^{\Sigma N}$ [18] and the *Dirac-Witten operator* $\hat{D}$ [26], respectively defined in a local o.n.b $(e_j)_{1\leq j\leq m}$ of TM by

$$D_M^{\Sigma N} := \sum_{j=1}^{m} e_j \underset{M}{\cdot} \nabla_{e_j}, \qquad \hat{D} := \sum_{j=1}^{m} e_j \cdot \tilde{\nabla}_{e_j}.$$

Both operators, which act on the sections of $\Sigma\tilde{M}_{|M}$, are elliptic, and from Lemma 3 are related by

$$\hat{D} = D_M^{\Sigma N} - \tfrac{1}{2}mH.$$

Furthermore, the operator $D_M^{\Sigma N}$ is formally self-adjoint (but $\hat{D}$ is *not*).

We now specialize to submanifolds with particular geometric structures. It is first important to point out that the spinor bundle ΣN is in general *not* isomorphic to ΣM; this may hold even if there exists an isomorphism between TM and NM, such as for Lagrangian submanifolds in Kählerian manifolds (see the examples in Notes 1). We therefore recall the notion of isomorphism between spin structures:

Definition 1. Let E and F be two spin vector bundles on a manifold M, with fixed spin structures $\mathrm{Spin}(E) \to \mathrm{SO}(E)$ and $\mathrm{Spin}(F) \to \mathrm{SO}(F)$. An *isomorphism* between the spin structures of E and F is given by a pair of principal-bundle isomorphisms $\mathrm{Spin}(E) \xrightarrow{\tilde{f}} \mathrm{Spin}(F)$ and $\mathrm{SO}(E) \xrightarrow{f} \mathrm{SO}(F)$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Spin}(E) \arrow[r, "\tilde{f}"] \arrow[d] & \mathrm{Spin}(F) \arrow[d] \arrow[dr] & \\
 & & M \\
\mathrm{SO}(E) \arrow[r, "f"] & \mathrm{SO}(F) \arrow[ur] &
\end{tikzcd}$$

If two vector bundles have isomorphic spin structures, they obviously have isomorphic spinor bundles as well. Hence we give the following

Corollary 4. *Assume that there exists an orientation-preserving isometry f from TM to NM which induces an isomorphism $(\tilde{f}, f)$ of the respective spin structures. Then there exists a complex-vector bundle isomorphism*

$$\Sigma\tilde{M}_{|M} \to \Lambda TM \otimes \mathbb{C} \tag{10}$$

satisfying:

- *For every tangent vector X to M and every ϕ in $\Sigma\tilde{M}_{|M}$, the element $X\cdot\phi$ is mapped onto $X\wedge\phi - X\lrcorner\phi$.*

- *For every vector* ν *in NM and every* ϕ *in* $\Sigma\tilde{M}_{|M}$, *the element* $\nu\cdot\phi$ *is mapped onto* $i\{f^{-1}(\nu)\wedge\phi+f^{-1}(\nu)\lrcorner\phi\}$.
- *If furthermore* f *is parallel w.r.t. the respective connections on TM and NM, then for every tangent vector field* X *to* M *and every section* ϕ *of* $\Sigma\tilde{M}_{|M}$, *the element* $\nabla_X\phi$ *is mapped onto* $\nabla^M_X\phi$.

Proof. The existence of (10) is a direct consequence of Corollary 3 and the fact that the spin structure of $T\tilde{M}_{|M}$ reduces via f to the spin structure of TM. The first property comes straigthforward. For the second one, it is to be noted that the automorphism

$$J:=\begin{pmatrix}0 & -f^{-1}\\ f & 0\end{pmatrix}$$

of $T\tilde{M}_{|M}$ is described through the Spin_n-reduction as

$$\mathrm{Spin}(TM)\times_{\mathrm{Ad}}\mathbb{R}^{2n}\to\mathrm{Spin}(TM)\times_{\mathrm{Ad}}\mathbb{R}^{2n},\qquad[\tilde{s},v]\mapsto[\tilde{s},J(v)].$$

The last statement follows from a short computation using the properties of compatibility between ∇ and the other objects on $\Sigma M\otimes\Sigma M(\oplus\Sigma M\otimes\Sigma M)$. □

Remark that, from the preceding proof, the existence of an orientation-preserving isometry $f:TM\to NM$ is equivalent to the existence of an almost-Hermitian structure J on $T\tilde{M}_{|M}$ mapping TM onto NM. Let $\tilde{\Omega}$ then denote the Kähler form of $(T\tilde{M}_{|M},g,J)$, i.e., $\tilde{\Omega}(X,Y):=g(J(X),Y)$ for all X and Y in $T\tilde{M}_{|M}$. Under the hypotheses of Corollary 4, the following holds: for every $0\le p\le n$ and ϕ in $\Lambda^pTM\otimes\mathbb{C}$,

$$\tilde{\Omega}\cdot\phi=i(2p-n)\phi$$

through (10). This also follows from the properties of (5), see previous subsection.

The existence of an almost-complex structure on $T\tilde{M}_{|M}$ is precisely the case we shall be interested in, since we shall consider submanifolds of Kählerian manifolds. We now recall the following.

Definition 2. A submanifold M^n of a Kählerian manifold $(\tilde{M}^{2n},g,J)$ is called *Lagrangian* if and only if

$$J(TM)=NM,$$

i.e., the complex structure identifies the tangent and normal bundles of the submanifold.

For a Lagrangian submanifold in a Kählerian manifold, the complex structure J obviously preserves the metric and the orientation of $T\tilde{M}_{|M}$, and is parallel.

Corollary 5. *Let* (M^n,g) *be a spin Lagrangian submanifold immersed in a Kählerian spin manifold* $(\tilde{M}^{2n},g,J)$. *Let the normal bundle NM carry the induced spin structure. Assume that the complex structure* J *induces an isomorphism between the spin structures of TM and NM. Then there exists a complex-vector bundle isomorphism*

$$\Sigma\tilde{M}_{|M}\to\Lambda TM\otimes\mathbb{C}$$

satisfying:

- *For every tangent vector X to M and every ϕ in $\Sigma\tilde{M}_{|M}$, the element $X \cdot \phi$ is mapped onto $X \wedge \phi - X \lrcorner \phi$.*
- *For every vector ν in NM and every ϕ in $\Sigma\tilde{M}_{|M}$, the element $\nu \cdot \phi$ is mapped onto $-i\{J(\nu) \wedge \phi + J(\nu) \lrcorner \phi\}$.*
- *For every tangent vector field X to M and every section ϕ of $\Sigma\tilde{M}_{|M}$, the element $\nabla_X \phi$ is mapped onto $\nabla^M_X \phi$.*

In particular, for each $0 \leq p \leq n$, the subspace $\Lambda^p TM \otimes \mathbb{C}$ is the eigenspace associated to the eigenvalue $i(2p - n)$ of the action of the Kähler form $\tilde{\Omega}$ of $(\tilde{M}^{2n}, g, J)$. Furthermore, for every section ϕ of $\Sigma\tilde{M}_{|M}$,

$$D_M^{\Sigma N} \phi = (d + \delta)\phi \quad \textit{and} \quad \hat{D}\phi = (d + \delta)\phi + \frac{im}{2}\{J(H) \wedge \phi + J(H) \lrcorner \phi\},$$

where d (resp. δ) denotes the exterior differential (resp. codifferential).

Proof. The only statement to be proved is the last one, for which it suffices to know that, for any local o.n.b. $(e_j)_{1 \leq j \leq n}$ of TM,

$$d = \sum_{j=1}^{n} e_j \wedge \nabla^M_{e_j} \quad \text{and} \quad \delta = -\sum_{j=1}^{n} e_j \lrcorner \nabla^M_{e_j}. \qquad \square$$

Notes 1.

(1) In the same way as above, one can give a "bundle-version" of Corollary 1: let E be any arbitrary Riemannian spin vector bundle on a spin manifold M such that there exists an isomorphism from TM to E, preserving the metric, the orientation *and the spin structure*. Then there exists a complex-vector bundle isomorphism between the Clifford bundle and the tensor product $\Sigma M \otimes \Sigma E$ (or double copy), mapping $X \cdot \phi$ onto $X \cdot_M \phi$ for every X in TM and ϕ in the Clifford bundle; if furthermore the isomorphism from TM to E is parallel w.r.t. the covariant derivatives on TM and E, then $\nabla^M_X \phi$ is mapped onto $\nabla^{\Sigma M \otimes \Sigma E}_X \phi$ (or double copy).

(2) We proved in Corollary 5 that, under a compatibility condition between the complex structure and the spin structures of TM and NM, the twisted-Dirac operator can be identified with $d+\delta$ (the so-called *Euler operator*), that is, a square-root of the Hodge–de Rham Laplacian $d\delta+\delta d$. This compatibility hypothesis is important, since otherwise the conclusions of Corollary 5 may fail as can be seen on the following example. Consider the unit circle $M := S^1$, canonically embedded in the complex line $\tilde{M} := \mathbb{C}$. This embedding is isometric and Lagrangian. Furthermore, S^1 carries two spin structures, a trivial one and a non-trivial one. If one chooses the trivial (resp. non-trivial) spin structure on the tangent bundle of S^1, then the induced spin structure on the normal bundle is non-trivial (resp. trivial) [3,7]. Therefore, the complex structure does not even preserve the spin bundles over S^1. Furthermore, the induced twisted-Dirac operator is in both cases the fundamental Dirac operator of S^1 for the non-trivial spin structure. Since this operator has trivial kernel, it cannot coincide with a square-root of the Hodge–de

Rham Laplacian. One therefore sees that the hypothesis of compatibility of Corollary 5 between the complex and the spin structures is necessary.

3. An upper eigenvalue bound for the twisted-Dirac operator on a Lagrangian submanifold

In this section, we consider a compact Lagrangian spin submanifold (M^n, g) in a Kählerian spin manifold $(\tilde{M}^{2n}, g, J)$. Since the operator $D_M^{\Sigma N}$ is elliptic and formally self-adjoint, it has a discrete spectrum; we then denote by λ_k ($k \in \mathbb{N} \setminus \{0\}$) its eigenvalues, counted with their multiplicities, assuming that $|\lambda_{k+1}| \geq |\lambda_k|$ for every $k \geq 1$.

We are interested in the following question: how can one control the smallest eigenvalues of the twisted-Dirac operator in terms of extrinsic geometric invariants? For submanifolds of certain real space-forms, it was proved by Bär in [2] and the author in [10,11] that the ambient curvature together with either the L^2 or the L^∞ norm of the mean curvature appear as the best candidates in that purpose. Those results were obtained considering restrictions to the submanifold of particular spinor fields on the ambient manifold, called *Killing spinors* (see [4] about those). As non Ricci-flat Kählerian spin manifolds of (real) dimension greater than 2 do not admit such spinor fields [14,19,20], it comes as a natural question whether such kind of estimates could still hold in our context. We give an affirmative and sharp answer to that problem, using the notion of *Kählerian Killing spinors* introduced by K.-D. Kirchberg in [17] and O. Hijazi in [15]. Remember that, for a complex constant α, an *α-Kählerian Killing spinor* on the Kählerian spin manifold $(\tilde{M}^{2n}, g, J)$ is a couple of sections (ψ, ϕ) of $\Sigma\tilde{M}$ satisfying, for every tangent vector field Z on $\tilde{M}$,

$$\tilde{\nabla}_Z\psi + \alpha p_-(Z)\cdot\phi = 0, \qquad \tilde{\nabla}_Z\phi + \alpha p_+(Z)\cdot\psi = 0,$$

where $p_\pm(Z) := (1/2)(Z \mp iJ(Z))$. When $\alpha = 0$, an α-Kählerian Killing spinor is just a pair of parallel spinor fields. As for Killing spinors, the presence of non-zero Kählerian Killing spinors yields strong conditions on the geometry of $\tilde{M}$ (see [15,17]): if $\alpha \neq 0$, the complex dimension n of $\tilde{M}$ has to be odd, the manifold $(\tilde{M}, g, J)$ has to be Einstein with scalar curvature $n(n+1)\alpha^2$ (therefore α must be either real or purely imaginary), and the sections ψ and ϕ have to lie in particular eigenspaces of the Clifford action of the Kähler form $\tilde{\Omega}$ of $(\tilde{M}, g, J)$:

$$\tilde{\Omega}\cdot\psi = -i\psi, \qquad \tilde{\Omega}\cdot\phi = i\phi$$

(remember that, in our convention, $\tilde{\Omega}(X, Y) := g(J(X), Y)$ for all vectors X and Y in $T\tilde{M}$). For example, the odd-complex-dimensional projective space $\mathbb{C}P^{2k+1}$ is a spin manifold carrying 1-Kählerian Killing spinors [16].

3.1. Main result

From here on, we denote by $\mathcal{K}(\alpha)$ the space of α-Kählerian Killing spinors on $(\tilde{M}, g, J)$ (note that, if $\alpha \neq 0$, then $\mathcal{K}(\alpha) \cap \mathcal{K}(-\alpha) = \{0\}$). Manifolds carrying a non-zero $\mathcal{K}(\alpha)$ have been completely characterized by Moroianu in [23] when α is a non-zero real number. The classification of spin Kählerian manifolds admitting a non-zero $\mathcal{K}(\alpha)$ with *purely imaginary* α is not known completely, but partial results have been obtained by Kirchberg in [17] and Herzlich in [13]. We prove the following:

Theorem 1. *Let (M^n, g) be a Lagrangian spin submanifold of a Kählerian spin manifold $(\tilde{M}^{2n}, g, J)$. Let the normal bundle of M in $\tilde{M}$ carry the induced spin structure, and H be the mean curvature vector field of M in $\tilde{M}$.*

Assume that, for a given non-zero complex constant α, the dimension of $\mathcal{K}(\alpha)$ is $N \geq 1$. Then the following holds:

(1) *The $2N^{\text{th}}$ eigenvalue λ_{2N} of the twisted-Dirac operator $D_M^{\Sigma N}$ satisfies*

$$(\lambda_{2N})^2 \leq \frac{1}{4}(n+1)^2\alpha^2 + \frac{1}{4}n^2||H||_\infty^2. \tag{11}$$

(2) *If furthermore α is a non-zero real number, the Nth eigenvalue λ_N satisfies*

$$(\lambda_N)^2 \leq \frac{(n+1)^2\alpha^2}{4} + \frac{n^2}{4\mathrm{Vol}(M)}\int_M |H|^2 v_g. \tag{12}$$

Proof. Let $\mathcal{K}_{(n-1)/2}$ (resp. $\mathcal{K}_{(n+1)/2}$) be the (pointwise) orthogonal projection of $\mathcal{K}(\alpha)$ onto the $-i$- (resp. i-) eigenspace of the Clifford action of $\tilde{\Omega}$, that is,

$$\mathcal{K}_{(n-1)/2} := \{\psi \in \Gamma(\Sigma\tilde{M}) / \tilde{\Omega}\cdot\psi = -i\psi \text{ and } \exists\phi \in \Gamma(\Sigma\tilde{M})/(\psi,\phi) \in \mathcal{K}(\alpha)\},$$
$$\mathcal{K}_{(n+1)/2} := \{\phi \in \Gamma(\Sigma\tilde{M}) / \tilde{\Omega}\cdot\phi = i\phi \text{ and } \exists\psi \in \Gamma(\Sigma\tilde{M})/(\psi,\phi) \in \mathcal{K}(\alpha)\}.$$

Since, from the hypotheses, $\alpha \neq 0$, the orthogonal projections $\mathcal{K}(\alpha) \to \mathcal{K}_{(n\pm1)/2}$ are injective. We therefore have $\dim_{\mathbb{C}}(\mathcal{K}_{(n\pm1)/2}) = \dim_{\mathbb{C}}(\mathcal{K}_{(n\pm1)/2}) = N$. We then give an upper bound of the Rayleigh-quotient

$$\mathcal{Q}((D_M^{\Sigma N})^2, \varphi) := \frac{\int_M \langle (D_M^{\Sigma N})^2\varphi, \varphi\rangle v_g}{\int_M \langle\varphi,\varphi\rangle v_g}$$

for $\varphi \in \mathcal{K}_{(n-1)/2} \oplus \mathcal{K}_{(n+1)/2}$, $\varphi \neq 0$, and apply the Min-Max principle.

Let (ψ, ϕ) be a non-zero α-Kählerian Killing spinor as above on $\tilde{M}$. To obtain $(D_M^{\Sigma N})^2\psi$ or $(D_M^{\Sigma N})^2\phi$, we first evaluate $\hat{D}^2$ on ψ or ϕ, then use the following relation ([10], Lemma 4.1): for every section φ of $\Sigma\tilde{M}_{|M}$ and in every local orthonormal basis (o.n.b.) $(e_j)_{1\leq j\leq n}$ of *TM*,

$$(D_M^{\Sigma N})^2\varphi = \hat{D}^2\varphi + \frac{n^2|H|^2}{4}\varphi + \frac{n}{2}\sum_{j=1}^n e_j \cdot \nabla_{e_j}^N H \cdot \varphi, \tag{13}$$

where $\nabla^N H$ denotes the normal covariant derivative of H.

Let us fix a local o.n.b. $(e_j)_{1\leq j\leq n}$ of *TM*. From the hypotheses,

$$\hat{D}\psi = \sum_{j=1}^n e_j \cdot \tilde{\nabla}_{e_j}\psi = -\alpha\sum_{j=1}^n e_j \cdot p_-(e_j)\cdot\phi.$$

For every vector X on $T\tilde{M}$, we have $g(p_-(X), p_-(X)) = 0$ and therefore $p_-(X)\cdot p_-(X)\cdot\varphi = 0$ for every φ in $\Sigma\tilde{M}_{|M}$. Hence

$$\hat{D}\psi = -\alpha\sum_{j=1}^{n} p_+(e_j)\cdot p_-(e_j)\cdot\phi.$$

But, since M is Lagrangian in $\tilde{M}$, the complex vectors $Z_j := p_+(e_j)$ and $\bar{Z}_j := p_-(e_j)$ ($1 \le j \le n$) form a Witt-basis for $T\tilde{M}\otimes\mathbb{C}$. Now remember the expression of the Kähler form $\tilde{\Omega}$ of $(\tilde{M}, g, J)$ in that basis:

$$\tilde{\Omega} = -2i\sum_{j=1}^{n} Z_j\wedge\bar{Z}_j.$$

We deduce from that identity that

$$\hat{D}\psi = -\alpha\sum_{j=1}^{n}(Z_j\wedge\bar{Z}_j)\cdot\phi + \alpha\sum_{j=1}^{n} g(Z_j,\bar{Z}_j)\phi = -\frac{i\alpha}{2}\tilde{\Omega}\cdot\phi + \frac{n\alpha}{2}\phi = \frac{(n+1)\alpha}{2}\phi,$$

since $\tilde{\Omega}\cdot\phi = i\phi$. A similar computation gives

$$\begin{aligned}\hat{D}\phi &= -\alpha\sum_{j=1}^{n} p_-(e_j)\cdot p_+(e_j)\cdot\psi = -\alpha\sum_{j=1}^{n}(\bar{Z}_j\wedge Z_j)\cdot\psi + \alpha\sum_{j=1}^{n} g(\bar{Z}_j, Z_j)\psi\\ &= \frac{i\alpha}{2}\tilde{\Omega}\cdot\psi + \frac{n\alpha}{2}\psi = \frac{(n+1)\alpha}{2}\psi,\end{aligned}$$

since $\tilde{\Omega}\cdot\psi = -i\psi$. We therefore obtain: $\hat{D}^2\psi = (n+1)^2\alpha^2/4\psi$ and $\hat{D}^2\phi = (n+1)^2\alpha^2/4\phi$, i.e.,

$$(D_M^{\Sigma N})^2\varphi = \frac{(n+1)^2\alpha^2}{4}\varphi + \frac{n^2|H|^2}{4}\varphi + \frac{n}{2}\sum_{j=1}^{n} e_j\cdot\nabla^N_{e_j}H\cdot\varphi \tag{14}$$

for $\varphi := \psi$ or ϕ, hence for every $\varphi\in\mathcal{K}_{(n-1)/2}\oplus\mathcal{K}_{(n+1)/2}$. Taking the Hermitian inner product of (14) with φ and integrating lead to

$$\mathcal{Q}((D_M^{\Sigma N})^2,\varphi) = \frac{(n+1)^2\alpha^2}{4} + \frac{n^2\int_M |H|^2\langle\varphi,\varphi\rangle v_g}{4\int_M\langle\varphi,\varphi\rangle v_g}. \tag{15}$$

Here we recall that, as the operator $D_M^{\Sigma N}$ is self-adjoint, we only keep the real parts when taking the Hermitian inner product of both members of (14) with φ. That is why the term involving $\nabla^N H$ does not give any contribution to (15).

We thereby obtain

$$\mathcal{Q}((D_M^{\Sigma N})^2,\varphi) \le \tfrac{1}{4}(n+1)^2\alpha^2 + \tfrac{1}{4}n^2||H||_\infty^2$$

for every $\varphi\in\mathcal{K}_{(n-1)/2}\oplus\mathcal{K}_{(n+1)/2}$. As the space $\mathcal{K}_{(n-1)/2}\oplus\mathcal{K}_{(n+1)/2}$ is $2N$-dimensional, the first statement straightforward follows from the Min-Max principle.

To conclude the second one, one just has to remember that, if furthermore α is real, then the length-function of $\psi + \phi$ is constant on $\tilde{M}$ (see [16]), hence on M. Summing up the identities (14) for $\varphi := \psi$ and $\varphi := \phi$ and integrating against $\psi + \phi$, one straightforward obtains the desired integral upper bound, but this time for the Nth eigenvalue only since $\mathcal{K}(\alpha)$ is N-dimensional. □

Note 1. If $\alpha = 0$, that is, if the ambient manifold $\tilde{M}$ admits non-trivial parallel spinors, then for every non-zero parallel spinor φ on $\tilde{M}$, we have $\hat{D}\varphi = 0$; it then follows from (13)) that

$$\mathcal{Q}((D_M^{\Sigma N})^2, \varphi) = \frac{n^2}{4\mathrm{Vol}(M)} \int_M |H|^2 v_g,$$

from which we directly obtain

$$(\lambda_N)^2 \leq \frac{n^2}{4\mathrm{Vol}(M)} \int_M |H|^2 v_g.$$

That estimate, which was proved by Bär in [2], could be included in Theorem 1 as the particular case $\alpha = 0$ of (12). We however point out that, for ambient manifolds carrying parallel spinors, we do not obtain any further information on the spectrum of $D_M^{\Sigma N}$ in presence of a Kähler structure on $\tilde{M}$.

Corollary 6. *Under the hypotheses of* Theorem 1, *assume furthermore that the complex structure J induces an isomorphism between the spin structures of TM and NM. Let H be the mean curvature vector field of M in $\tilde{M}$. Then the following holds*:

(1) *The $2N$ smallest eigenvalues (counted with their multiplicities) λ of the Hodge–de Rham Laplacian on $\Omega^{(n-1)/2}(M) \oplus \Omega^{(n+1)/2}(M)$ satisfy*

$$\lambda \leq \tfrac{1}{4}(n+1)^2\alpha^2 + \tfrac{1}{4}n^2||H||_\infty^2.$$

(2) *If furthermore α is a non-zero real number, the N smallest eigenvalues λ satisfy*

$$\lambda \leq \frac{(n+1)^2\alpha^2}{4} + \frac{n^2}{4\mathrm{Vol}(M)} \int_M |H|^2 v_g.$$

If moreover M is *minimal in $\tilde{M}$ (i.e., if $H = 0$), then the same result hold for the N (resp. $[(N+1)/2]$) smallest eigenvalues of the Hodge–de Rham Laplacian on the space of closed $(n+1)/2$-forms.*

Proof. From Corollary 5, if J identifies the spin structures of TM and NM, then $(D_M^{\Sigma N})^2 = d\delta + \delta d$. Furthermore, the isomorphism (10) identifies the eigenspace associated to the eigenvalue $i(2p-n)$ of the Clifford action of $\tilde{\Omega}$ with $\Lambda^p TM \otimes \mathbb{C}$; since, under that action, the spinor field ϕ (resp. ψ) is eigen for the eigenvalue i (resp. $-i$), it is a $(n+1)/2$-form (resp. a $(n-1)/2$-form) on M. Hence the first statement holds. If moreover $H = 0$, then

$\hat{D} = D_M^{\Sigma N} = d + \delta$. From the equalities $\hat{D}\psi = ((n+1)\alpha/2)\phi$ and $\hat{D}\phi = ((n+1)\alpha/2)\psi$, we then deduce that

$$\left| \begin{array}{l} d\psi = \dfrac{(n+1)\alpha}{2}\phi \\ \delta\psi = 0 \end{array} \right. \quad \text{and} \quad \left| \begin{array}{l} d\phi = 0 \\ \delta\phi = \dfrac{(n+1)\alpha}{2}\psi, \end{array} \right.$$

i.e., ψ is coclosed and ϕ is closed. As the spectrum of the $(n-1)/2$-Laplacian on coclosed forms coincides with that of the $(n+1)/2$-Laplacian on closed forms (use the Hodge star operator), we obtain the second property. □

3.2. Examples

For an odd integer $n \geq 3$, consider the round sphere S^n (of constant sectional curvature 1) of dimension n as canonically embedded in the $2n+1$-dimensional round sphere S^{2n+1}. That embedding is isometric, totally geodesic, and the canonical complex structure of $\mathbb{R}^{2n+2}$ maps the tangent bundle of S^n into the horizontal space $\mathcal{H}$ defined, for each z in S^{2n+1} as

$$\mathcal{H}_z := \{\mathbb{R}z \oplus \mathbb{R}Jz\}^\perp \subset T_z S^{2n+1}$$

with the following property:

$$\mathcal{H}_{|S^n} = TS^n \underset{\perp}{\oplus} J(TS^n).$$

Let then $\mathbb{C}\mathrm{P}^n$ be the complex projective space of complex dimension n. Composing the Hopf fibration $S^{2n+1} \to \mathbb{C}\mathrm{P}^n$ with the above embedding yields an immersion

$$S^n \to \mathbb{C}\mathrm{P}^n \tag{16}$$

satisfying the following: it is isometric (the Hopf fibration induces an isometry from $\mathcal{H}$ onto $T\mathbb{C}\mathrm{P}^n$), totally geodesic (the Hopf fibration maps horizontal geodesics onto geodesics) and Lagrangian (the Hopf fibration is "holomorphic" w.r.t. the complex structures of $\mathcal{H}$ and $\mathbb{C}\mathrm{P}^n$). Furthermore, if n is odd, the manifold $\mathbb{C}\mathrm{P}^n$ is spin, has a unique spin structure since it is simply-connected [18], and carries a $2C_n^{(n+1)/2}$-dimensional space of 1-Kählerian Killing spinors [16] (remember that $C_n^p := n!/p!(n-p)!$). The round sphere S^n is also spin, and for the same reason has a unique spin structure; more generally, every spin vector bundle on S^n has a unique spin structure, that is, two spin structures on a vector bundle on S^n will always be isomorphic.

Consider then the (canonical) spin structure of TS^n and the induced one on the normal bundle of S^n in $\mathbb{C}\mathrm{P}^n$ w.r.t. (16); then the complex structure of $\mathbb{C}\mathrm{P}^n$ will necessarily induce an isomorphism between the spin structures of the tangent and normal bundles of S^n. Hence we obtain from Corollary 6 the existence of the following upper bound for the $2C_n^{(n+1)/2}$ smallest eigenvalues λ of the Hodge–de Rham Laplacian on the closed $(n+1)/2$-forms:

$$\lambda \leq \tfrac{1}{4}(n+1)^2.$$

That estimate is sharp: indeed, for $1 \leq p \leq n-1$, the spectrum of the Hodge–de Rham Laplacian on the closed p-forms on S^n is [8]

$$\{(k+p)(n-p+k+1)/k \in \mathbb{N}\},$$

and the multiplicity of the first eigenvalue ($k = 0$) is C^p_{n+1}. But, for $p := (n + 1)/2$, we have $2C_n^{(n+1)/2} = C_n^{(n-1)/2} + C_n^{(n+1)/2} = C_{n+1}^{(n+1)/2}$, which is precisely the multiplicity of the first eigenvalue of the Hodge–de Rham Laplacian on the closed $(n + 1)/2$-forms.

A further interesting example would be to consider the real n-dimensional projective space (with $n = 4k + 3$) in the complex projective space $\mathbb{C}\mathrm{P}^n$. That question, which is linked to determining *all* the Lagrangian submanifolds which satisfy the equality in Theorem 1, will be considered in a forthcoming work.

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References

[1] C. Bär, Harmonic spinors for twisted Dirac operators, Math. Ann. 309 (1997) 225–246.
[2] C. Bär, Extrinsic bounds for eigenvalues of the Dirac operator, Ann. Glob. Anal. Geom. 16 (1998) 573–596.
[3] C. Bär, Dirac eigenvalues: the case of compact manifolds, to appear in the Proceedings of the Workshop Dirac Operators: Yesterday and Today, American University of Beirut, International Press.
[4] H. Baum, T. Friedrich, R. Grunewald, I. Kath, Twistor and Killing Spinors on Riemannian Manifolds, Teubner-Texte zur Mathematik, vol. 124, Teubner-Verlag, Stuttgart/Leipzig, 1991.
[5] J.-P. Bourguignon, O. Hijazi, J.-L. Milhorat, A. Moroianu, A Spinorial approach to Riemannian and conformal geometry, in preparation.
[6] B.-Y. Chen, On the first eigenvalue of Laplacian of compact minimal submanifolds of rank one symmetric spaces, Chin. J. Math. 11 (4) (1983) 1–15.
[7] T. Friedrich, Dirac operators in Riemannian geometry, in: Graduate Studies in Mathematics, vol. 25, American Mathematical Society, 2000.
[8] S. Gallot, D. Meyer, Opérateur de courbure et laplacien des formes différentielles d'une variétériemannienne, J. Math. Pure Appl. IX. Sér. 54 (1975) 259–284.
[9] P. Gauduchon, Private communications.
[10] N. Ginoux, Opérateurs de Dirac sur les sous-variétés, Thèse de doctorat, Université Henri Poincaré, Nancy, 2002.
[11] N. Ginoux, Une nouvelle estimation extrinsèque du spectre de l'opérateur de Dirac, C. R. Acad. Sci. Paris Sér. I 336 (10) (2003) 829–832.
[12] N. Ginoux, B. Morel, On eigenvalue estimates for the submanifold Dirac operator, Int. J. Math. 13 (5) (2002) 533–548.
[13] M. Herzlich, Scalar curvature and rigidity of odd-dimensional complex hyperbolic spaces, Math. Ann. 312 (1998) 641–657.
[14] O. Hijazi, A conformal lower bound for the smallest eigenvalue of the Dirac operator and Killing spinors, Commun. Math. Phys. 104 (1986) 151–162.
[15] O. Hijazi, Eigenvalues of the Dirac operator on compact Kähler manifolds, Commun. Math. Phys. 160 (3) (1994) 563–579.

[16] K.-D. Kirchberg, The first eigenvalue of the Dirac operator on Kähler manifolds, J. Geom. Phys. 7 (4) (1990) 449–468.
[17] K.-D. Kirchberg, Killing spinors on Kähler manifolds, Ann. Global Anal. Geom. 11 (2) (1993) 141–164.
[18] H.B. Lawson, M.-L. Michelsohn, Spin Geometry, Princeton University Press, 1989.
[19] A. Lichnerowicz, Killing spinors according to O. Hijazi and applications, in: Spinors in Physics and Geometry (Trieste 1986), World Scientific Publishing, Singapore, 1988.
[20] A. Lichnerowicz, Spin manifolds, killing spinors and the universality of the Hijazi inequality, Lett. Math. Phys. 3 (1987) 331–344.
[21] J. Milnor, Remarks concerning spin manifolds, Princeton University Press, 1965, pp. 55–62.
[22] B. Morel, Eigenvalue estimates for the Dirac-Schrödinger operators, J. Geom. Phys. 38 (2001) 1–18.
[23] A. Moroianu, La première valeur propre de l'opérateur de Dirac sur les variétés kählériennes compactes, Commun. Math. Phys. 169 (1995) 373–384.
[24] A. Ros, Spectral geometry of CR-minimal submanifolds in the complex projective space, Kodai Math. J. 6 (1) (1983) 88–99.
[25] J. Tits, Tabellen zu den einfachen Lie Gruppen und ihren Darstellungen, Lecture Notes in Math, vol. 40, Springer, 1967.
[26] E. Witten, A new proof of the positive energy theorem, Commun. Math. Phys. 80 (1981) 381–402.